RESEARCHERID
THOMSON REUTERS

[Raj, 5(12): December2018] ISSN 2348 – 8034
DOI- 10.5281/zenodo.2526366 Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

EXISTENCE OF BEST PROXIMITY POINTS USING ASYMPTOTIC CENTER

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Abstract

Fixed point theorems for a nonexpansive mappings via the concept of asymptotic center studied in many literature. In this manuscript, we introduce the notion of asymptotic center of two sequences $\{x_n\}$,$\{y_n\}$ and discuss some properties of it. Also, as an application of this concept best proximity point theorem for certain mapping is proved.

AMS Subject Classification: 54H25, 47H10

Keywords: *Asymptotic center, Fixed points, Best proximity points, Relatively Nonexpansive mapping.*

I. INTRODUCTION

Let X be a Banach space. A mapping $T: X \to X$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$. A well-known fixed point theorem due to Browder, Gohde and Kirk states that every nonexpansive mapping on a closed bounded convex subset of a uniformly convex Banach space into itself has at least one fixed point. Existence of a fixed point for such mappings are studied in many literature. The following interesting method for studying fixed point properties of nonexpansive mappings via the concept of asymptotic center was first introduced by M. Edelstein. Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. Define the asymptotic radius of $\{x_n\}$ at x as a number $r(x, \{x_n\}) = \lim_{n\to\infty} \sup\|x - x_n\|$. The asymptotic radius of $\{x_n\}$ in K is defined as the infimum of $r(x, \{x_n\})$ over K. A point z is said to be an asymptotic center of the sequence $\{x_n\}$ in K if $r(z, x_n) = \inf\{r(x, \{x_n\}): x \in K\}$. Also using this concept, the author proved the following fixed point theorem which guarantees the existence of a fixed point for a mapping which is weaker than the nonexpansive mapping.

Theorem 1.1. Let $f: K \to K$ be a mapping of a closed convex subset of a uniformly convex Banach space X into itself and let$\{f^n(x): \mathrm{n} \in \mathbb{N}\}$be a bounded sequence of iterates of some x in K having the asymptotic center c in K. If there is an $\mathrm{N} \in \mathbb{N}$ exists such that

$$\|f^n(x) - f(\mathrm{c})\| \leq \|f^{\mathrm{n}-1}(x) - \mathrm{c}\|, \mathrm{n} > N$$

Then the asymptotic center c is a fixed point of f.

Let us consider two nonempty subsets A, B of a Banach space X and a mapping $T: A \to B$. It is clear that the necessary condition for the existence of a fixed point for T is $T(A) \cap A \neq \emptyset$. If the fixed point equation $Tx = x$ does not possess a solution then $d(x, Tx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. A point $x_0 \in A$ is called a best proximity of T if $d(x_0, Tx_0) = \mathrm{dist}$ (A, B). Note that $\mathrm{dist}(A, B) = 0$, then the best proximity point is nothing but a fixed point of T.

On the other hand, Eldred and et.al. introduced a new class of mapping called relatively nonexpansive mapping and they proved the best proximity point theorem for such mappings.

Definition 1. Let A, B be a nonempty subsets of a Banach space X. A cyclic mapping $\mathrm{T}: \mathrm{A} \cup \mathrm{B} \to \mathrm{A} \cup \mathrm{B}$ is said to be relatively nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \text{for all } x \in A, y \in B$$

In this paper, we introduce the notion of asymptotic center of two sequences $\{x_n\}, \{y_n\}$ and discuss some properties of it. Also we discuss the existence of an optimal solution of the class of mapping which is weaker than the class of relatively nonexpansive mapping.

II. PRELIMINARIES

In this section, we discuss some notations and known results.
Let X be a Banach space and K be a nonempty subset of X. The asymptotic radius of a sequence defines a functional $r: X \to \mathbb{R}^+$ by

$$r(x,\{x_n\}) = \lim_{n\to\infty} \sup\|x - x_n\|.$$

The asymptotic center z of a bounded sequence $\{x_n\}$ with respect to$K$ is defined by

$$r(z,\{x_n\}) = r(K,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\}$$

The set of all asymptotic centers of $\{x_n\}$ with respect to K is denoted by $Z(K,\{x_n\})$.

Remark 1.

1. The functional r is convex.(i.e., $r(\alpha x + \beta y,\{x_n\}) \le \alpha\, r(x,\{x_n\}) + \beta\, r(y,\{x_n\})$
2. If K is convex then $Z(K,\{x_n\})$ is convex.

Now we discuss the existence and uniqueness of asymptotic centers of bounded sequences. First we define the modulus of convexity and characteristic of convexity of a Banach space.

Definition 2. Let X be a Banach Space. Then a function $\delta_X: [0,2] \to [0,1]$ is said to be the modulus of convexity of X if

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}$$

It is easy to see that $\delta_X(0) = 0$ and δ_X is increasing function.

Definition 3. The characteristic of convexity or the coefficient of convexity of a Banach space X is the number

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_x(\epsilon) = 0\}$$

where $\delta(\varepsilon)$ denotes the modulus of convexity of a Banach space.

The Banach space X is said to be uniformly convex if $\varepsilon_0(x) = 0$. Also, the diameter of a set K by diam $(K) = \sup\{\|x - y\| : x, y \in K\}$.

Theorem 2.1. Let K be a nonempty closed convex subset of a Banach space$X$and$\{x_n\}$a bounded sequence in X. Then $\mathrm{diam}(Z(K,\{x_n\}) \le \varepsilon_0(X)\, r(K,\{x_n\})$.
The following theorem guarantees the uniqueness of an asymptotic center of a Banach space.

Theorem 2.2. Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in X.Then $Z(K,\{x_n\})$ is a singleton set.

Using the uniqueness of asymptotic center the following fixed point theorem proved by M. Edelstein.

Theorem 2.3. Let $f : K \to K$ be a mapping of a closed convex subset of a uniformly convex Banach space X into itself and let $\{f_n(x) : n \in \mathbb{N}\}$ be a bounded sequence of iterates of some x in K having the asymptotic center c in K. If an N exists such that

$$\|f^n(x) - f(c)\| \le \|f^{n-1}(x) - c\|, n > N.$$

Then $f(c) = c$.
Next we define the notion of P- property which plays a vital role in our main results.

Definition4. Let A, B be nonempty subsets of a Banach space X. Then the pair (A, B) is said to have P-property if and only if,

$\| x_1 - y_1 \| = \mathrm{dist}(A,B)$ and $\| x_2 - y_2 \| = \mathrm{dist}(A,B)$ then $\| x_1 - x_2 \| = \| y_1 - y_2 \|$.

In [7], It has been shown that X is a strictly convex Banach space if and only if every pair (A,B) of closed convex subsets of X has P−property.

III. MAIN RESULTS

In this section, we define asymptotic center for two sequences and using this definition we provided the sufficient conditions for existence of best proximity points for a class of mapping.

Definition 5. Let A, B be a nonempty subset of X and $\{x_n\} \subseteq A, \{y_n\} \subseteq B$ be two sequences. Consider the functional $r: A \times B \to \mathbb{R}^+$ defined by,

$$r(x, y) = \max\{\limsup \|x - y_n\|, \limsup \|y - x_n\|\}$$

$$r(A, B) = \inf_{x \in A, y \in B} r(x, y)$$

A point $(x_0, y_0) \in (A, B)$ with $\|x_0 - y_0\| = \operatorname{dist}(A, B)$is said to be asymptotic center if $r(x_0, y_0) = r(A, B)$. The collection of all asymptotic centers of $\{x_n\}, \{y_n\}$ denoted by $Z(A, B)$.

Remark 2.

(1) The functional r is convex.

Proof. Let $r(x_1, y_1) \& r(x_2, y_2) \in (A, B)$

$$\begin{aligned} r(\alpha(x_1, y_1) + \beta(x_2, y_2)) &= r(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\ &= \max\{\limsup\|\alpha x_1 + \beta x_2 - y_n\|, \limsup\|\alpha y_1 + \beta y_2 - x_n\| \\ &\leq \max\{\limsup \alpha k\|x_1 - y_n\| + \limsup\beta \|x_2 - y_n\|, \\ &\qquad \limsup \alpha\|y_1 - x_n\| + \limsup\beta\|y_2 - x_n\|\} \\ &\leq \max\{\alpha r(x_1, y_1) + \beta r(x_2, y_2), \alpha r(x_1, y_1) + \beta r(x_2, y_2)\} \\ &= \alpha r(x_1, y_1) + \beta r(x_2, y_2) \end{aligned}$$

(2) $Z(A, B)$is convex if A, B is convex.

Proof. Let (x_1, y_1) and $(x_2, y_2) \in Z(A, B)$.
Therefore $r(x_1, y_1) = r(A, B) = r(x_2, y_2)$.
By Remark 1,$r\big(t(x_1, y_1) + (1 - t)(x_2, y_2)\big) \leq tr(x_1, y_1) + (1 - t)r(x_2, y_2)$

$$= tr(A, B) + (1 - t)r(A, B) = r(A, B).$$

Hence $t(x_1, y_1) + (1 - t)(x_2, y_2) \in Z(A, B)$

(3) The function$r: A \times B \to \mathbb{R}^+$ is Continuous.

Let us define the diameter of two subsets. Let M, N be non-empty subsets of Banach Space. Then $\operatorname{diam}(M, N) = \operatorname{Sup}\{\|m - n\|: m \in M, n \in N\}$

Theorem 3.1. LetA, Bbe closed convex subset of a Banach space$X$and$\{x_n\}, \{y_n\}$be bounded sequence in A, Brespectively.Then $\operatorname{diam}(Z(A, B)) \leq \varepsilon_0(X) r(A, B) + \operatorname{dist}(A, B)$

Proof. Let $d = \operatorname{diam}\big(Z(A, B)\big)$
If $Z(A, B)$ is empty then nothing to prove.
If $Z(A, B)$ is singleton then there exists $(x_0, y_0) \in Z(A, B)$ with $\|x_0 - y_0\| = \operatorname{dist}(A, B)$ then the result is obvious.
If $(x_1, y_1)$and$(x_2, y_2) \in Z(A, B)$. For $0 < r_1 < d$ such that $\|x_2 - y_1\| \geq d - r_1$. Now

$$r(x_1, y_1) = r(A, B) = \max\{\limsup\|x_1 - y_n\|, \limsup\|y_1 - x_n\|\}$$

$$r(x_2, y_2) = r(A, B) = \max\{\limsup\|x_2 - y_n\|, \limsup\|y_2 - x_n\|\}$$

Since$Z(A, B)$ is convex, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \in Z(A, B)$

Already we have

(1) $\limsup\|x_1 - y_n\| \leq r(A, B)$

(2) $\limsup\|x_2 - y_n\| \leq r(A, B)$

(3) $\limsup\|x_1 - y_n\| \leq r(A,B)$
(4) $\limsup\|x_2 - y_n\| \leq r(A,B)$.

Now $\limsup\left\|y_n - \left(\frac{x_1+x_2}{2}\right)\right\| \leq \left(1-\delta\left(\frac{\|x_1-x_2\|}{r(A,B)}\right)\right)r(A,B)$ and

$\limsup\left\|x_n - \left(\frac{y_1+y_2}{2}\right)\right\| \leq \left(1-\delta\left(\frac{\|y_1-y_2\|}{r(A,B)}\right)\right)r(A,B)$.

Since $\|x_2 - x_1\| \geq \|x_2 - y_1\| - \|x_1 - y_1\| \geq (d - r_1) - \text{dist}(A,B)$ and

$$\|y_1 - y_2\| \geq \|y_1 - x_2\| - \|x_2 - y_2\| \geq (d - r_1) - \text{dist}(A,B)$$

We have $-\delta\left(\frac{\|y_1-y_2\|}{r(A,B)}\right) \leq -\delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right)$

Similarly proceeding we have $-\delta\left(\frac{\|x_1-x_2\|}{r(A,B)}\right) \leq -\delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right)$

Therefore $r(A,B) \leq \left(1-\delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right)\right)r(A,B)$

$$\Rightarrow 1-\delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right) \geq 1$$

$\Rightarrow \delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right) = 0$.By the definition of $\varepsilon_0(X)$,

$$\delta\left(\frac{(d-r_1)-\text{dist}(A,B)}{r(A,B)}\right) \leq \varepsilon_0(X) \Rightarrow (d-r_1) \leq \varepsilon_0(X)r(A,B) + \text{dist}(A,B)$$

Since r_1 is arbitrary,$\text{diam}(Z(A,B)) \leq \varepsilon_0(X)r(A,B) + \text{dist}(A,B)$.

LEMMA 3.1. Let A, Bbe a nonempty closed convex subset of a reflexive Banach space$X$ and $f: A \times B \to [-\infty,\infty]$ is proper lower semi continuous convex function such that $f(x_n, y_n) \to \infty$ as $\|x_n\| \to \infty$ or $\|y_n\| \to \infty$. Then there exists$x_0 \in A, y_0 \in B \ni f(x_0, y_0) = \inf\{f(x,y) : x \in A, y \in B\}$.

*Proof.*Let $m = \inf\{f(x,y): x \in A, y \in B\}$. Choose a minimizing sequence $\{x_n\} \subseteq A$and $\{y_n\} \subseteq B \ni f(x_n, y_n) \to \infty$. If $\{x_n\}$ or $\{y_n\}$ is not bounded, there exist a subsequence $\{x_{n_i}\}$ or $\{y_{n_i}\} \ni \|x_{n_i}\| \to \infty$ or $\|y_{n_i}\| \to \infty$. By assumption $f(x_{n_i}, y_{n_i}) \to \infty$ a contradiction to $m \neq \infty$. So $\{y_n\}$and$\{x_n\}$is bounded. Since X is reflexive, $\exists$ a subsequence $x_{n_j} \rightharpoonup x_0, y_{n_j} \rightharpoonup y_0$.

$$m \leq f(x_0, y_0) \leq \lim inf\, f(x_{n_j}, y_{n_j}) = \lim f(x_n, y_n) = m.$$

Theorem 3.2. Let A, B be a nonempty closed convex subset of a uniformly convex Banach space $X$and$\{x_n\},\{y_n\}$be a sequence inA, B.Then $Z(A,B)$ is singleton set.

Proof. By previous lemma, $Z(A,B) \neq \emptyset$. For the uniformly convex Banach space $\varepsilon_0(X) = 0$ therefore by previous theorem, $\text{diam}(Z(A,B)) = \text{dist}(A,B)$. Suppose $Z(A,B)$ is not a singleton set, there exists $(x_0,y_0),(x_1,y_1) \in Z(A,B)$ and $\text{diam}(Z(A,B)) = \text{dist}(A,B)$ we get contradiction to P- property. Hence $Z(A,B)$ is singleton set.

By using the uniqueness of asymptotic center for two sequences we prove the following best proximity point theorem for a mapping assuming less on relatively nonexpansive mapping.

Theorem 3.3. Let A, B be closed convex subset of a uniformly convex Banach space X. $f : A \cup B \to A \cup B$ be cyclic mapping and let $\{f^{2n}(x)\}$ and $\{f^{2n+1}(x)\}$ be bounded sequence of iterates for some $x \in A$ has an asymptotic center $(c, c') \in A \times B$. If there is an $N \in \mathbb{N}$ such that $\|f^{2n+2}(x) - f(c)\| \leq \|f^{2n+1}(x) - c\|, n > N$.
Then f has a best proximtity point.i.e.,there exists z such that $\|z - f(z)\| = \text{dist}(A,B)$.

Proof.

$$\begin{aligned} r(c, f(c)) &= \max\{\limsup\|f^{2n}(x) - f(c)\|, \limsup\|f^{2n+1}(x) - c\|\} \\ &= \max\{\limsup\|f^{2n+2}(x) - f(c)\|, \limsup\|f^{2n+1}(x) - c\|\} \\ &= \limsup\|f^{2n+1}(x) - c\| \\ &\leq \max\{\limsup\|f^{2n+1}(x) - c\|, \limsup\|f^{2n}(x) - c'\|\} \end{aligned}$$

$$= r(c, c')$$

By above theorem, $f(c) = c'$. Therefore from the definition of asymptotic center $\|c - f(c)\| = \|c - c'\| = \text{dist}(A, B)$.

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